2839 [1920, 274].

By translating the steps of the construction of a regular pentagon from plane geometry into algebra show that one of the fifth roots of unity is equal to

$$\tfrac{1}{4}(\sqrt{5}-1)+\tfrac{1}{4}i\sqrt{10+2\sqrt{5}}.$$

(This problem is proposed for solution in Wilczynski and Slaught, *College Algebra with Applications*, Boston, 1916, p. 193.)

I. Solution by H. L. Olson, University of Michigan.

Consider a circle with center at the origin, O, and unit radius, intersecting the positive x-axis at A and the negative y-axis at D. With center at the mid-point, E, of OD, draw a circular arc passing through A and intersecting the positive y-axis at F. The radius of this circle is $\frac{1}{2}\sqrt{5}$, and hence $\overline{OF} \doteq \frac{1}{2}(\sqrt{5}-1)$. With center A and radius AF, draw a circular arc intersecting the circle with center O at G and H. AG and AH are two sides of a regular pentagon inscribed in the circle O, and $\overline{OG}$ and $\overline{OH}$ are the Argand representations of two of the fifth roots of unity.

Since $AF^2 = 1 + (\frac{1}{2}\sqrt{5}-\frac{1}{2})^2$, the equations of the circles O and A are

$$x^2+y^2=1,$$
$$(x-1)^2+y^4 = 1+(\tfrac{1}{2}\sqrt{5}-\tfrac{1}{2})^2.$$

Hence, the coördinates of G and H are

$$x=\tfrac{1}{4}(\sqrt{5}-1), \qquad y=\pm\tfrac{1}{4}\sqrt{10+2\sqrt{5}};$$

and two of the fifths roots of unity are $\frac{1}{4}[(\sqrt{5}-1)\pm i\sqrt{10+2\sqrt{5}}]$.

II. Solution by Otto Dunkel, Washington University.

In texts on geometry a pentagon is usually constructed by dividing the radius OA, here taken as of unit length, at M in extreme and mean ratio. The length OM is laid off twice on the circle as chords giving the points A, K, G; then AK is the side of a regular decagon and AG is the side of a regular pentagon. From the definition of extreme and mean ratio it follows that $OM^2 = OA\cdot MA = 1 - OM$, and hence $OM = \frac{1}{2}(\sqrt{5}-1)$. It easily follows that $MG=OG=1$ and hence the coördinates of G are $x=\frac{1}{4}(\sqrt{5}-1)$, $y=\sqrt{1-x^2}=\frac{1}{4}\sqrt{10+2\sqrt{5}}$. Hence the complex number represented by G is as stated in the problem.

Also solved by T. M. Blakslee, Arthur Pelletier, A. V. Richardson, C. H. Richardson, and F. L. Wilmer.

2853 [1920, 377]. Proposed by J. S. BROWN, Southwest Texas State Normal College, San Marcos, Texas.

Find the side and apothem of a regular pentagon inscribed in a circle without the use of extreme and mean ratio.

Three Solutions by T. M. Blakslee, Ames, Iowa.

I. Let $x+iy$ be the point on the unit circle whose angle is 36°. Then the side of a regular inscribed pentagon will be $p=2y$. In the equation $(x+iy)^5=-1$, the coefficient of i is

$$y(5x^4-10x^2y^2+y^4)=0.$$

We wish the smaller of the two positive values of y. Therefore we can remove the factor y and if we substitute $1-y^2$ for x^2, our equation reduces to

$$16y^4-20y^2+5=0.$$

Whence $y^2=(10-2\sqrt{5})/16$, $p=\frac{1}{2}\sqrt{10-2\sqrt{5}}$, and the apothem is $a=\frac{1}{4}(\sqrt{5}+1)$.